*The Non-homogeneous Differential Equation of Parabolic Type.**

By Griffith C. Evans.

1. *Introduction.*

It is characteristic of the partial differential equation of parabolic type that on one side it partakes much of the nature of the equation of elliptic type, and on the other, that of hyperbolic type. To be more precise, if we are considering an equation in two variables, x and t, the solution is something like an harmonic function with respect to the variable x, in regard to which the differentiation is of the second order, but more like a solution of an equation of hyperbolic type with respect to the variable t. For instance, if in the equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=f(x,t) \tag{1}$$

the function $f(x,t)$ is analytic in the variable x in the neighborhood of the point (x_1, t_1), every solution u of equation (1), which is continuous together with its derivatives $\partial u/\partial x$, $\partial^2 u/\partial x^2$, $\partial u/\partial t$, is an analytic function of x in that same neighborhood.† We know, moreover, that a regular solution of equation (1) exists, under suitable boundary conditions, provided that $f(x,t)$ and its derivative in regard to x are finite and continuous.‡ In the corresponding equation of elliptic type a solution exists if the function with its first derivatives with respect to both variables are finite and continuous (Gauss).

We know that the second-order derivatives of the definite integral which represents the potential function exist in the neighborhood of a point P_1, if the density ρ at any point P in that neighborhood satisfies the condition

$$|\rho-\rho_1|\leq A r^{\nu},$$

where A is a constant, r is the distance from P_1 to P, and ν is a positive number

* Presented to the American Mathematical Society, September, 1914.

† E. E. Levi, "Sull' equazione del calore," *Annali di Matematica*, Vol. XIV (1907–1908), p. 239.

‡ W. A. Hurwitz, "Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung," p. 85, Göttingen, 1910.

not zero (Kronecker). This is not a necessary condition, of course, but it is one which is convenient in actual use. If we look for a similar condition with respect to the definite integral u, which is used to represent the principal solution of equation (1), and ask when the derivatives $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ exist, we find Levi's theorem that there are regular solutions of (1), under given boundary conditions, provided that in the neighborhood of any point (x_1, t_1), $f(x, t)$ satisfies a condition of the form

$$\left| \frac{f(x, t) - f(x_1, t_1)}{[(x-x_1)^2 + (t-t_1)^2]^\nu} \right| < N.^*$$

If we write the expression analogous to the potential function, for equation (1), the derivatives of that quantity exist in the neighborhood of (x_1, t_1) if the function $f(x, t)$ is finite and integrable in the given region, and at (x_1, t_1) satisfies Levi's condition. But this condition is symmetrical in x and t, and therefore misses in part the essential nature of the parabolic equation. We are thus led to look for an asymmetrical condition more analogous to Kronecker's but not so much like it.

2. *The Principal Solution of Equation* (1).

Let us consider a region ${}_{t_0}R_{t_1}$ (notation of Hurwitz) bounded on the left by the line $t = t_0$, on the right by the line $t = t_1$, above by the curve $x = \xi_2(t)$ and below by the curve $x = \xi_1(t)$. The functions ξ_1 and ξ_2 are to be continuous with their first derivatives, and are to have only a finite number of maxima and minima in the interval under consideration. Moreover, we are to have $\xi_2(t) > \xi_1(t)$, for $t \geq t_0$.

We shall say that the function $u(x, t)$ is regular at a point (x_0, t_0) if u and $\partial u/\partial x$ are finite and continuous in the neighborhood of (x_0, t_0), and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ exist at (x_0, t_0). We shall say that $u(x, t)$ is regular in a region ${}_{t_0}R_{t_1}$ if it is regular at every point inside the region, continuous with its derivative $\partial u/\partial x$ on the boundary, and if $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are linearly integrable in regard to x and t respectively. In this paper we shall not deal extensively with regularity at a point; we shall, however, touch on the matter again, in the last section.

Making use of the notation (Levi)

$$h_{\alpha,\beta}(x, t \mid x_1, t_1) = \frac{(x-x_1)^\alpha}{(t-t_1)^\beta} e^{-\frac{(x-x_1)^2}{4(t-t_1)}}, \tag{2}$$

* E. E. Levi, *loc. cit.*, p. 239.

we have the following theorem for the existence of the principal solution of equation (1).

THEOREM I. *The function*

$$u(x,t)=\frac{1}{2\sqrt{\pi}}\iint_{({}_{t_0}R_{t_1})} h_{0,\frac{1}{2}}(\xi,\tau\,|\,x,t)\,f(\xi,\tau)\,d\xi\,d\tau \tag{3}$$

is regular in the neighborhood of the point (x_1, t_1), *and satisfies there the equation* (1), *if* $f(x,t)$ *is finite and integrable in* ${}_{t_0}R_{t_2}$ $(t_2>t_1)$, *is continuous in the neighborhood of* (x_1, t_1), *and satisfies the condition*

$$|f(x,t)-f(x',t)|\leq A\,|x-x'|^{\nu}, \tag{4}$$

where (x,t) *and* (x',t') *are any two points in that neighborhood, and* A *and* ν *are positive constants, not zero.*

3. *The Functions* u *and* $\partial u/\partial x$.

From the finiteness and integrability of $f(x,t)$ follows at once the continuity of u and the existence and continuity of $\partial u/\partial x$, within and on the boundary of R; for we may differentiate under the integral sign once in regard to x.* But we may not differentiate twice in regard to x or once in regard to t, under the integral sign, and we must therefore establish the existence of these derivatives in other ways.

4. *The Function* $\partial^2 u/\partial x^2$.

Let us consider now the point (x_1, t_1) within R, and divide the region R into two parts, of which one shall be a small open rectangle ${}_a r_t$, wholly within the above-mentioned neighborhood of (x_1, t_1), and the other the rest of ${}_{t_0}R_t$. If we separate the integral in (3) into the corresponding two parts, denoting by u' the result of extending the integral over r, and by u'' the result due to the rest of R, we shall have for $\partial^2 u''/\partial x^2$ a continuous function of x and t. It remains then to investigate u'.

For this purpose let us write

$$\left.\begin{aligned} f(x,t) &= f(x',t)+\phi(x,t,x'),\\ u'(x,t) &= u_1(x,t,x')+u_2(x,t,x'),\\ u_1(x,t,x') &= \frac{1}{2\sqrt{\pi}}\iint_{({}_a r_t)} h_{0,\frac{1}{2}}(\xi,\tau\,|\,x,t)\,f(x',\tau)\,d\xi\,d\tau, \end{aligned}\right\} \tag{5}$$

thus defining the functions $\phi(x,t,x')$, $u_1(x,t,x')$, $u_2(x,t,x')$. We shall have, then,

* E. E. Levi, *loc. cit.*, p. 234.

$$u_2(x, t, t') = \frac{1}{2\sqrt{\pi}} \iint_{(a^{r}t)} h_{0, \frac{1}{2}}(\xi, \tau \mid x, t)\, \phi(\xi, \tau, x')\, d\xi\, d\tau. \tag{6}$$

To u_1 we may apply the theorem of Hurwitz already mentioned, from which it follows that if the function $f(x, t)$ is finite and integrable in R, and continuous at (x_1, t), with its first derivative in regard to x, then the function $u(x, t)$, defined by (3), is regular at and in the neighborhood of (x_1, t_1). Therefore, since the derivative, with respect to x, of $f(x', t)$ is continuously zero, the function u_1, defined by (5), is regular at (x', t'). In particular, we have

$$\frac{\partial^2 u_1(x, t, x')}{\partial x^2} = \frac{1}{4\pi} \int_a^{t'} [h_{1, \frac{3}{2}}(\xi_2(\tau), \tau \mid x, t) - h_{1, \frac{3}{2}}(\xi_1(\tau), \tau \mid x, t)]\, f(x', \tau)\, d\tau, \tag{7}$$

a function which inside r is continuous in x, t and x'.*

Let us now actually calculate $\partial^2 u_2/\partial x^2 = \lim\limits_{\Delta x = 0} \Delta[\partial u_2/\partial x]/\Delta x$, when $x = x'$ and $t = t'$. To do this we may follow closely the analysis of E. E. Levi,† replacing his condition by ours, and obtain the result

$$\left(\frac{\partial^2 u_2(x, t, x')}{\partial x^2}\right)_{\substack{x=x'\\ t=t'}} = -\frac{1}{2\pi} \iint_{(a^{r}t)} h_{0, \frac{3}{2}}(\xi, \tau \mid x', t')\, \phi(\xi, \tau, x')\, d\xi\, d\tau$$
$$+ \frac{1}{4\pi} \iint_{(a^{r}t)} h_{2, \frac{5}{2}}(\xi, \tau \mid x', t')\, \phi(\xi, \tau, x')\, d\xi\, d\tau. \tag{8}$$

The presence of the ϕ makes the above integrals convergent. Each of them, moreover, represents a continuous function of x' and t' in the neighborhood of (x_1, t_1), as we see by dividing r into two parts, one of them being a very small neighborhood of (x_1, t_1). For, if we form the expression (8) for (x'', t''), and subtract from it its value for (x', t'), keeping both points well within the very small neighborhood, we can make the contributions due to this neighborhood as small as we please, on account of the uniformity of the condition (4); then we can make the rest of the expression as small as we please by taking the point (x'', t'') close enough to the point (x', t').

We see from the results just obtained that the function

$$\left(\frac{\partial^2 u'(x, t)}{dx^2}\right)_{x=x',\, t=t'} = \left(\frac{\partial^2 u_1(x, t, x')}{\partial x^2}\right)_{x=x',\, t=t'} + \left(\frac{\partial^2 u_2(x, t, x')}{\partial x^2}\right)_{x=x',\, t=t'} \tag{9}$$

represents a continuous function of x' and t' in the neighborhood of the point (x_1, t_1), and is therefore linearly integrable in regard to x' and t' in that neighborhood. Accordingly it follows that we have established those same properties for the function $\partial^2 u/\partial x^2$.

---

* W. A. Hurwitz, *loc. cit.*, p. 86.

† E. E. Levi, *loc. cit.*, p. 236.

5. *The Function $\partial u/\partial t$.*

We might in the same way, by means of slightly more complicated expressions, investigate the function $\partial u/\partial t$. It is more convenient, however, to make use of a different method, which is more general, and adopts an entirely different point of view. To do this we shall need to mention one or two introductory definitions and theorems.

A *standard curve* is a closed curve composed of a finite number of pieces, which does not cut itself at any point; along each piece the coordinates of a point are given by two functions $\phi(q)$ and $\psi(q)$, continuous with their first derivatives, throughout a finite interval for q. In this interval $\phi'(q)$ and $\psi'(q)$ shall not vanish together; and neither of them shall vanish at more than a finite number of points unless it vanishes throughout the whole interval. We see, then, that a standard curve can not be cut by a vertical or horizontal straight line in more than a finite number of points unless the straight line includes itself a portion of the curve.

We shall say that a standard curve approaches a point uniformly if, when we are given a circle with center at the point and radius arbitrarily small, the standard curve becomes and remains entirely within the given circle.

We have the following general theorem: *If, in the x, y plane, s is a standard curve, enclosing a region σ, which approaches a point P uniformly, and the functions u and $\partial u/\partial x$ are continuous throughout the neighborhood of P, then*

$$\frac{\partial u}{\partial x} = \lim \frac{1}{\sigma} [\textstyle\int_s u\, dy]. \tag{10}$$

On the other hand, if merely u is continuous throughout the neighborhood of P, and lim $[\int_s u\, dy]/\sigma$ exists when σ approaches P uniformly, then $\partial u/\partial x$ exists at P and is equal to that limit.

A similar theorem applies, of course, to

$$\frac{\partial u}{\partial y} = -\lim \frac{1}{\sigma} [\textstyle\int_s u\, dx]. \tag{11}$$

The first part of this theorem is proved by integrating $\partial u/\partial x$ over the region σ, and the second part may be obtained by choosing for s a rectangle whose side Δy is conveniently small with reference to its side Δx.

Besides this general theorem, we need the following result which applies particularly to the parabolic equation: *If $f(x, t)$ is finite and continuous within the region ${}_{t_0}R_t$, the function $u(x, t)$, defined by (3), is continuous with its derivative in regard to x within and on the boundary of the region, and satisfies the equation*

$$\int_s\left[u(x,t)\,dx+\frac{\partial u(x,t)}{\partial x}\,dt\right]=\int\int_\sigma f(x,t)\,dx\,dt, \tag{12}$$

where s is any standard curve entirely within the region (the axis of t being horizontal, as heretofore).* From this theorem it follows that if $f(x,t)$ is finite and integrable in R and continuous in the neighborhood of a point P within R, and s is a standard curve which approaches P uniformly, then

$$\lim\frac{1}{\sigma}\left[\int_s\left[u(x,t)\,dx+\frac{\partial u(x,t)}{\partial x}\,dt\right]\right]=f(x,t). \tag{13}$$

Let us apply these results to our present problem. Since as we have already shown, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ exist and are continuous throughout the neighborhood of (x_1, y_1), it follows that as s approaches P uniformly, P being in that neighborhood, we have the relation

$$\frac{\partial^2 u}{\partial x^2}=-\lim\frac{1}{\sigma}\left[\int_s\frac{\partial u}{\partial x}\,dt\right]. \tag{14}$$

Hence, from (13), $\lim\,[\int_s u\,dx]/\sigma$ exists as s approaches P uniformly, and, since u is continuous in the neighborhood of P, our general theorem tells us that $\partial u/\partial t$ exists at P and is given by the equation

$$\frac{\partial u}{\partial t}=\lim\frac{1}{\sigma}\,[\int_s u\,dx]. \tag{15}$$

If now we substitute (14) and (15) in (13), the equation (13) becomes merely the differential equation (1), and since by that equation $\partial u/\partial t$ may be written, in the neighborhood of (x_1, t_1), as the sum of two continuous functions of x and t, it is itself continuous in that neighborhood. Theorem I is therefore completely established.

6. *Regularity at a Point.*

Regularity throughout a region is necessary if we wish to apply Green's theorem. But if we are interested only in regularity at a point, we can get less restrictive conditions for it than those expressed in Theorem I. A sufficient condition for regularity at a point is given by the following theorem:

THEOREM II. *The function given by equation* (3) *is regular at the point* (x_1, t_1), *and satisfies there the equation* (1), *if* $f(x,t)$ *is finite and integrable*

* G. C. Evans, "On the Reduction of Integrodifferential Equations," *Transactions of the American Mathematical Society*, Vol. XV (1914), p. 477.

in ${}_{t_0}R_{t_2}$ $(t_2 > t_1)$, *and in the neighborhood of* (x_1, t_1) *is continuous and satisfies the condition*

$$|f(x, t) - f(x_1, t)| \leq N|x - x_1|^{\nu}, \tag{4'}$$

where N aud ν *are positive constants, not zero.*

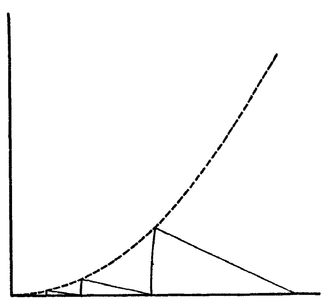
This theorem may be established directly by calculating $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ at the point (x_1, t_1). Or section 5 may be applied, provided that the curves s be throughout restricted to rectangles with sides parallel to the axes, and the general theorem there mentioned be extended to the following: If, along the line $x = x_1$, in the neighborhood of the point (x_1, y_1), we have, uniformly,

$$\lim_{\Delta x = 0} \frac{u(x_1 + \Delta x, y) - u(x_1, y)}{\Delta x} = \frac{\partial u}{\partial x},$$

then at (x_1, y_1), if s approaches (x_1, y_1) uniformly, we have

$$\frac{\partial u}{\partial x} = \lim \frac{1}{\sigma} [\textstyle\int_s u\, dy].$$

The condition (4′) might hold for every point of the region, and the condition (4) still fail to hold. Similarly, Levi's condition might hold for every point of a region,* and conditions (4′) and (4) still fail to hold, although for most purposes these conditions are more general than Levi's. In other words, Levi's condition and the condition (4′) are such that they do not necessarily hold uniformly even if they hold at every point. An example will make this clear.



Consider a function $f(x)$, defined as follows: Divide the interval (0, 1) by the points 1/2, 1/4, 1/8, ..., and in each sub-interval draw the curve $y = 2(x - p)^p$, where p is 1/2, 1/4, 1/8, ..., connecting the point where each of these curves cuts the curve $y = x^2$ to the next point of division to the right with a straight line. Let $f(x)$ be the curve made up of these portions

* In the development referred to above, Levi considers only regularity at a point. If Levi's condition is made uniform, his theorem becomes a particular case of Theorem I.

of straight lines and curves $y=2(x-p)^{\nu}$, when $x>0$; when $x\leq 0$, let $f(x)=0$. We notice now that for every point in the interval $0\leq x\leq 1$ we have a condition of the form

$$|f(x)-f(x_1)|\leq N|x-x_1|^{\nu},$$

where N and ν are positive constants, not zero, depending in value on the value of x_1. We can not find a positive ν which will hold for all points of the interval, because if we let x_1 approach 0 by coinciding with the points 1/2, 1/4, 1/8, ..., the value of ν approaches 0.

THE RICE INSTITUTE, *July*, 1914.